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Free products of dimonoids

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Abstract. We construct a free product of dimonoids which generalizes a free dimonoid presented by J.-L. Loday and describe its structure.

1. Introduction

A *dimonoid* is a nonempty set D equipped with two binary associative operations $\circ$ and $*$ satisfying the axioms $(x \circ y) \circ z = x \circ (y * z)$, $(x * y) \circ z = x * (y \circ z)$, $(x \circ y) * z = x * (y * z)$, while a *dialgebra* is just a linear analog of a dimonoid. These notions were introduced by J.-L. Loday [1] for solving of problems in Leibniz algebras and investigated by many authors (see, e.g., [2]).

The construction of a free dimonoid generated by a given set was presented by J.-L. Loday [1] and applied to the study of free dialgebras and a cohomology of dialgebras. Structural properties of free dimonoids have been investigated in [3].

In this paper we present a construction of a free product of arbitrary dimonoids which generalizes a free dimonoid and describe its structure. The obtained results extend the corresponding results from [1] and [3].

2. The main result

As usual, we denote the set of all positive integers by $\mathbb{N}$.

Let $Fr[S_i]_{i\in I}$ be the free product of arbitrary semigroups S_i, $i \in I$. For every $w \in Fr[S_i]_{i\in I}$ denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$) and the length of w by l_w. Consider the set

$$G(S_i)_{i\in I} = \{(w, m) \in Fr[S_i]_{i\in I} \times \mathbb{N} \,|\, l_w \geqslant m\}.$$

For all $(w, m) \in G(S_i)_{i\in I}$ and $u \in Fr[S_i]_{i\in I}$ assume

$$f^u_{(w,m)} = \begin{cases} l_u + m, & l_{u^{(1)}w^{(0)}} = 2, \\ l_u + m - 1, & l_{u^{(1)}w^{(0)}} = 1. \end{cases} \tag{1}$$

We need the following two lemmas.

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Lemma 2.1. *Let (u,s), $(\omega,s)\in G(S_i)_{i\in I}$ and $w\in Fr[S_i]_{i\in I}$. If $u^{(0)},\omega^{(0)}\in S_i$ for some $i\in I$, then $f^w_{(u,s)}=f^w_{(\omega,s)}$.* $\square$

Lemma 2.2. *For all $w_1,w_2,w_3\in Fr[S_i]_{i\in I}$ and $(w_3,m_3)\in G(S_i)_{i\in I}$,*

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=f^{w_1w_2}_{(w_3,m_3)}.$$

Proof. As $(w_2w_3)^{(0)}\in S_i\Leftrightarrow w_2^{(0)}\in S_i$ for some $i\in I$ and $(w_1w_2)^{(1)}\in S_j\Leftrightarrow w_2^{(1)}\in S_j$ for some $j\in I$, then

$$l_{w_1{}^{(1)}(w_2w_3)^{(0)}}=l_{w_1^{(1)}w_2^{(0)}},\tag{2}$$

$$l_{(w_1w_2)^{(1)}w_3^{(0)}}=l_{w_2^{(1)}w_3^{(0)}}.\tag{3}$$

It is not difficult to see that

$$l_{w_1w_2}=\begin{cases} l_{w_1}+l_{w_2},\ l_{w_1^{(1)}w_2^{(0)}}=2,\\ l_{w_1}+l_{w_2}-1,\ l_{w_1^{(1)}w_2^{(0)}}=1.\end{cases}\tag{4}$$

By (1) – (3) we have

$$f^{w_2}_{(w_3,m_3)}=\begin{cases} l_{w_2}+m_3,\ l_{w_2^{(1)}w_3^{(0)}}=2,\\ l_{w_2}+m_3-1,\ l_{w_2^{(1)}w_3^{(0)}}=1,\end{cases}\tag{5}$$

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=\begin{cases} l_{w_1}+f^{w_2}_{(w_3,m_3)},\ l_{w_1^{(1)}w_2^{(0)}}=2,\\ l_{w_1}+f^{w_2}_{(w_3,m_3)}-1,\ l_{w_1^{(1)}w_2^{(0)}}=1,\end{cases}\tag{6}$$

$$f^{w_1w_2}_{(w_3,m_3)}=\begin{cases} l_{w_1w_2}+m_3,\ l_{w_2^{(1)}w_3^{(0)}}=2,\\ l_{w_1w_2}+m_3-1,\ l_{w_2^{(1)}w_3^{(0)}}=1.\end{cases}\tag{7}$$

Further, using (4) – (7), consider the following four cases.

Case 1. $l_{w_1^{(1)}w_2^{(0)}}=l_{w_2^{(1)}w_3^{(0)}}=2$. Then $f^{w_2}_{(w_3,m_3)}=l_{w_2}+m_3$ and

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=l_{w_1}+f^{w_2}_{(w_3,m_3)}=l_{w_1}+l_{w_2}+m_3=l_{w_1w_2}+m_3=f^{w_1w_2}_{(w_3,m_3)}.$$

Case 2. $l_{w_1^{(1)}w_2^{(0)}}=2$ and $l_{w_2^{(1)}w_3^{(0)}}=1$. Then $f^{w_2}_{(w_3,m_3)}=l_{w_2}+m_3-1$ and

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=l_{w_1}+f^{w_2}_{(w_3,m_3)}=l_{w_1}+l_{w_2}+m_3-1=l_{w_1w_2}+m_3-1=f^{w_1w_2}_{(w_3,m_3)}.$$

Case 3. $l_{w_1^{(1)}w_2^{(0)}}=1$ and $l_{w_2^{(1)}w_3^{(0)}}=2$. Then $f^{w_2}_{(w_3,m_3)}=l_{w_2}+m_3$ and

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=l_{w_1}+f^{w_2}_{(w_3,m_3)}-1=l_{w_1}+l_{w_2}+m_3-1=l_{w_1w_2}+m_3=f^{w_1w_2}_{(w_3,m_3)}.$$

Case 4. $l_{w_1^{(1)}w_2^{(0)}}=l_{w_2^{(1)}w_3^{(0)}}=1$. Then $f^{w_2}_{(w_3,m_3)}=l_{w_2}+m_3-1$ and

$$f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=l_{w_1}+f^{w_2}_{(w_3,m_3)}-1=l_{w_1}+l_{w_2}+m_3-2=l_{w_1w_2}+m_3-1=f^{w_1w_2}_{(w_3,m_3)}.$$

Thus, $f^{w_1}_{(w_2w_3,\ f^{w_2}_{(w_3,m_3)})}=f^{w_1w_2}_{(w_3,m_3)}$. $\square$

For a given relation ρ on a dimonoid $(D, \circ, *)$, the congruence generated by ρ is the least congruence on $(D, \circ, *)$ containing ρ. It will be denoted by $\rho^\star$ and can be characterized as the intersection of all congruences on $(D, \circ, *)$ containing ρ.

Let $\{(D_i, \circ_i, *_i)\}_{i\in I}$ be a family of arbitrary pairwise disjoint dimonoids. Operations on $Fr[(D_i, \circ_i)]_{i\in I}$ and $Fr[(D_i, *_i)]_{i\in I}$ will be denoted by $\circ$ and $*$ respectively. For every $i \in I$ consider a relation $\theta_i = \{(a *_i b, a \circ_i b) \mid a, b \in D_i\}$ on a dimonoid $(D_i, \circ_i, *_i)$. It is clear that operations of $(D_i, \circ_i, *_i)/\theta_i^\star$ coincide and it is a semigroup.

Let $\omega_1 = (x_1 x_2 \dots x_s, t)$, $\omega_2 = (y_1 y_2 \dots y_p, r) \in G((D_i, \circ_i))_{i\in I}$, where elements $x_1, \dots, x_s, y_1, \dots, y_p \in \bigcup_{i\in I} D_i$. Define a relation $\sim$ on $G((D_i, \circ_i))_{i\in I}$ by putting

$$\omega_1 \sim \omega_2 \Leftrightarrow \begin{cases} s = p,\ t = r \text{ and } x_k \theta_{j_k}^\star y_k \text{ for all } 1 \leqslant k \leqslant s \text{ and some } j_k \in I, \\ \text{at that } x_t = y_r. \end{cases}$$

It is not difficult to check that $\sim$ is an equivalence relation. Denote by $[w, m]$ the equivalence class of $\sim$ containing an element $(w, m) \in G((D_i, \circ_i))_{i\in I}$, and by $G^\star((D_i, \circ_i))_{i\in I}$ the quotient set $G((D_i, \circ_i))_{i\in I}/\sim$.

Observe that (1) does not depend on the definition of operations on semigroups S_i, $i \in I$, and define operations $\circ'$ and $*'$ on $G^\star((D_i, \circ_i))_{i\in I}$ by

$$[w_1, m_1] \circ' [w_2, m_2] = [w_1 \circ w_2,\ m_1],$$

$$[w_1, m_1] *' [w_2, m_2] = [w_1 * w_2,\ f^{w_1}_{(w_2, m_2)}]$$

for all $[w_1, m_1], [w_2, m_2] \in G^\star((D_i, \circ_i))_{i\in I}$. The algebra $(G^\star((D_i, \circ_i))_{i\in I}, \circ', *')$ will be denoted by $\breve{G}(D_i)_{i\in I}$.

Theorem 2.3. *$\breve{G}(D_i)_{i\in I}$ is the free product of dimonoids $(D_i, \circ_i, *_i), i \in I$.*

Proof. First note that from the associativity of the operation of a free product semigroups it follows that

$$(w_1 \circ w_2) \circ w_3 = w_1 \circ (w_2 \circ w_3), \tag{8}$$

$$(w_1 * w_2) \circ w_3 = w_1 * (w_2 \circ w_3), \tag{9}$$

$$(w_1 * w_2) * w_3 = w_1 * (w_2 * w_3) \tag{10}$$

for all $w_1, w_2, w_3 \in Fr[(D_i, \circ_i)]_{i\in I}$.

The associativity of the operation $\circ'$ follows from (8). Let further $[w_1, m_1]$, $[w_2, m_2]$, $[w_3, m_3] \in \breve{G}(D_i)_{i\in I}$. Then

$$\begin{aligned} ([w_1, m_1] \circ' [w_2, m_2]) \circ' [w_3, m_3] &= [w_1 \circ w_2, m_1] \circ' [w_3, m_3] \\ &= [(w_1 \circ w_2) \circ w_3, m_1] = [w_1 \circ (w_2 \circ w_3), m_1] \\ &= [w_1 \circ (w_2 * w_3), m_1] = [w_1, m_1] \circ' [w_2 * w_3,\ f^{w_2}_{(w_3, m_3)}] \\ &= [w_1, m_1] \circ' ([w_2, m_2] *' [w_3, m_3]), \end{aligned}$$

by (8) and the condition $(w_1 \circ (w_2 \circ w_3), m_1) \sim (w_1 \circ (w_2 * w_3), m_1)$.

Moreover, using (9) and Lemma 2.1, we obtain

$$\begin{aligned}([w_1,m_1]*'[w_2,m_2])\circ'[w_3,m_3] &= [w_1*w_2,\ f^{w_1}_{(w_2,m_2)}]\circ'[w_3,m_3]\\ &= [(w_1*w_2)\circ w_3,\ f^{w_1}_{(w_2,m_2)}]\\ &= [w_1*(w_2\circ w_3),\ f^{w_1}_{(w_2\circ w_3,m_2)}]\\ &= [w_1,m_1]*'[w_2\circ w_3,m_2]\\ &= [w_1,m_1]*'([w_2,m_2]\circ'[w_3,m_3]).\end{aligned}$$

Further we get

$$\begin{aligned}[w_1,m_1]*'([w_2,m_2]*'[w_3,m_3]) &= [w_1,m_1]*'[w_2*w_3,\ f^{w_2}_{(w_3,m_3)}]\\ &= [w_1*(w_2*w_3),\ f^{w_1}_{(w_2*w_3,\ f^{w_2}_{(w_3,m_3)})}]\\ &= [(w_1*w_2)*w_3, f^{w_1\circ w_2}_{(w_3,m_3)}] = [(w_1\circ w_2)*w_3, f^{w_1\circ w_2}_{(w_3,m_3)}]\\ &= [w_1\circ w_2,m_1]*'[w_3,m_3]\\ &= ([w_1,m_1]\circ'[w_2,m_2])*'[w_3,m_3]\end{aligned}$$

and

$$\begin{aligned}([w_1,m_1]*'[w_2,m_2])*'[w_3,m_3] &= [w_1*w_2,\ f^{w_1}_{(w_2,m_2)}]*'[w_3,m_3]\\ &= [(w_1*w_2)*w_3,\ f^{w_1*w_2}_{(w_3,m_3)}],\end{aligned}$$

according to (10), Lemma 2.2 and the fact that

$$((w_1*w_2)*w_3, f^{w_1\circ w_2}_{(w_3,m_3)}) \sim ((w_1\circ w_2)*w_3, f^{w_1\circ w_2}_{(w_3,m_3)}).$$

This shows that $\breve{G}(D_i)_{i\in I}$ is a dimonoid. Moreover, for each $(D_i,\circ_i,*_i), i\in I$, we have

$$(D_i,\circ_i,*_i)\cong \widetilde{D}_i = \{[w,1]\in \breve{G}(D_i)_{i\in I} \mid w\in D_i\}$$

and all subdimonoids $\widetilde{D}_i, i\in I$, generate $\breve{G}(D_i)_{i\in I}$.

In order to complete the proof we should check the condition of continuability of a homomorphism. For this let

$$\alpha_i : (D_i,\circ_i,*_i)\to (T,\circ'',*''),$$

where $i\in I$, be a homomorphism from $(D_i,\circ_i,*_i)$ to an arbitrary dimonoid $(T,\circ'',*'')$. Define a map

$$\alpha : \breve{G}(D_i)_{i\in I}\to (T,\circ'',*'') : [x_1\dots x_k\dots x_s,t]\mapsto [x_1\dots x_k\dots x_s,t]\alpha,$$

assuming

$$[x_1\dots x_k\dots x_s,t]\alpha = x_1\gamma_1 *'' \dots *'' x_t\gamma_t \circ'' \dots \circ'' x_s\gamma_s,$$

where $\gamma_k=\alpha_k$ for $x_k\in D_k$, $1\leqslant k\leqslant s$.

A straightforward verification shows that $[w_1,m_1]\alpha=[w_2,m_2]\alpha$ for all $[w_1,m_1]$, $[w_2,m_2]\in \breve{G}(D_i)_{i\in I}$, if $(w_1,m_1)\in[w_2,m_2]$, and so, α is well-defined.

Using axioms of a dimonoid and homomorphisms $\alpha_i, i\in I$, one can show that α is a homomorphism continuing $\alpha_i, i\in I$. Thus, $\breve{G}(D_i)_{i\in I}$ is the free product of dimonoids $(D_i,\circ_i,*_i), i\in I$. □

From Theorem 2.3 we obtain

Corollary 2.4. *The free dimonoid is the free product* $\breve{G}(D_i)_{i\in I}$ *of one-generated free dimonoids* $(D_i, \circ_i, *_i), i \in I$.

Proof. Observe that the free dimonoid $(D(X), \circ, *)$ of an arbitrary rank was constructed in ([1], p. 15) and the structure of one-generated free dimonoids was described in [3]. By Lemma 3 from [3] $\breve{G}(D_i)_{i\in I} \cong (D(X), \circ, *)$. □

3. The structure of $\breve{G}(D_i)_{i\in I}$

Let $B(I)$ be the semilattice of all nonempty finite subsets of I with respect to the operation of the set theoretical union. For every $w = x_1 x_2 \dots x_l \dots x_k \in Fr[(D_i, \circ_i)]_{i\in I}$ assume $\tilde{c}(w) = \bigcup_{l=1}^{k} \{x_l j'\}$, where

$$j' : \bigcup_{i\in I} D_i \to I : a \mapsto i, \text{ if } a \in D_i,\ i \in I.$$

For every $Y \in B(I)$ and all $x, y \in Y$ let

$$H_Y = \{[w, m] \in \breve{G}(D_i)_{i\in I} \mid \tilde{c}(w) = Y\},$$

$$H_Y^{(x,y)} = \{[w, m] \in H_Y \mid (w^{(0)} j', w^{(1)} j') = (x, y)\},$$

$Y \times Y$ be a rectangular band, that is, a semigroup with the operation $(x, y)(a, b) = (x, b)$. It is easy to see that H_Y is a subdimonoid of $\breve{G}(D_i)_{i\in I}$ and $H_Y^{(x,y)}$ is a subdimonoid of H_Y.

In terms of dibands of subdimonoids (see, e.g., [4]) we obtain the following structure theorem.

Theorem 3.1. *The free product* $\breve{G}(D_i)_{i\in I}$ *of dimonoids* $(D_i, \circ_i, *_i), i \in I$, *is a semilattice* $B(I)$ *of subdimonoids* $H_Y, Y \in B(I)$. *Every dimonoid* $H_Y, Y \in B(I)$, *is a rectangular band* $Y \times Y$ *of subdimonoids* $H_Y^{(x,y)}$, $(x, y) \in Y \times Y$.

Proof. Assuming

$$c' : [w, m] \mapsto \tilde{c}(w),$$

we obtain a homomorphism from $\breve{G}(D_i)_{i\in I}$ to $B(I)$ as

$$\tilde{c}(w \star u) = \tilde{c}(w) \cup \tilde{c}(u)$$

for all $w, u \in Fr[(D_i, \circ_i)]_{i\in I}$ and $\star \in \{\circ, *\}$. Hence, $\breve{G}(D_i)_{i\in I}$ is a semilattice $B(I)$ of subdimonoids $H_Y, Y \in B(I)$.

Now we shall prove the second part of the theorem.

Let

$$\pi : H_Y \to Y \times Y : [w, m] \mapsto (w^{(0)} j', w^{(1)} j').$$

As

$$(w \star u)^{(0)} j' = w^{(0)} j', \quad (w \star u)^{(1)} j' = u^{(1)} j'$$

for all $w, u \in Fr[(D_i, \circ_i)]_{i \in I}$ and $\star \in \{\circ, *\}$, then π is a homomorphism. From here, H_Y is a rectangular band $Y \times Y$ of subdimonoids $H_Y^{(x,y)}$, $(x, y) \in Y \times Y$. $\square$

We finish this section with the description of some congruence on $\breve{G}(D_i)_{i \in I}$ when $\circ_i = *_i$ for all $i \in I$.

First observe that if $\circ_i = *_i$ for all $i \in I$, then $\sim$ is the diagonal of $G(S_i)_{i \in I}$ and $G(S_i)_{i \in I}/\sim$ is identified with $G(S_i)_{i \in I}$. It is clear that in this case $\circ = *$.

Let α be an arbitrary fixed congruence on $Fr[S_i]_{i \in I}$. Define a relation $\tilde{\alpha}$ on $\breve{G}(S_i)_{i \in I}$ by

$$[w_1, m_1] \tilde{\alpha} [w_2, m_2] \Leftrightarrow w_1 \alpha w_2$$

for all $[w_1, m_1], [w_2, m_2] \in \breve{G}(S_i)_{i \in I}$.

It is not difficult to prove the following lemma.

Lemma 3.2. *The relation $\tilde{\alpha}$ is a congruence on the dimonoid $\breve{G}(S_i)_{i \in I}$ and operations of the quotient dimonoid $\breve{G}(S_i)_{i \in I}/\tilde{\alpha}$ coincide.*

From Lemma 3.2 we obtain

Corollary 3.3. *If α is the diagonal of $Fr[S_i]_{i \in I}$, then $\breve{G}(S_i)_{i \in I}/\tilde{\alpha}$ is the free product of semigroups.*

Note that Theorem 3.1, Lemma 3.2 and Corollary 3.3 extend, respectively, Theorem 3, Lemma 5 and Corollary 1 from [3].

References

[1] **J. L. Loday**, *Dialgebras*, Lect. Notes Math. **1763**, Springer-Verlag, Berlin (2001), 7 – 66.

[2] **A. P. Pozhidaev**, *Dialgebras and related triple systems*, Sib. Math. J. **49** (2008), 696 – 708.

[3] **A. V. Zhuchok**, *Free dimonoids*, Ukr. Math. J. **63** (2011), 196 – 208.

[4] **A. V. Zhuchok**, *Dimonoids*, Algebra and Logic **50** (2011), 323 – 340.

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