BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(71), 2013, Pages 130–134
ISSN 1024–7696

Semilattice decompositions of trioids

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Abstract. We describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of s-simple subtrioids.

Mathematics subject classification: 17A30, 20M10.
Keywords and phrases: Trioid, semilattice congruence, semilattice of subtrioids, dimonoid, semigroup.

1 Introduction

Trioids were introduced by J.-L. Loday and M. O. Ronco [1] for the study of ternary planar trees. Trialgebras, which are based on the notion of a trioid, have been studied in different papers (see, for example, [1–3]). It is well known that the notion of a trioid generalizes the notion of a dimonoid [4, 5]. Dimonoids play a prominent role in problems from the theory of Leibniz algebras. Trioids were studied in some papers of the author (see, for example, [6–8]). Note that if the operations of a trioid coincide then it becomes a semigroup. So, trioids are a generalization of semigroups.

In this work we describe semilattice decompositions of trioids. In Section 2 we give necessary definitions, auxiliary results (Proposition 1 and Lemma 1) and describe some connections between trioids and dimonoids (Lemma 2). Yamada [9] described all semilattice congruences on an arbitrary semigroup and proved that every semigroup is a semilattice of s-simple semigroups. These results were generalized to dimonoids in [10]. In Section 3 we extend results from [10] to the case of trioids (Theorems 1 and 2).

2 Preliminaries

A nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$ and $\perp$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T5}$$

$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T6}$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T7}$$

$$(x \perp y) \vdash z = x \vdash (y \vdash z) \tag{T8}$$

for all $x, y, z \in T$, is called a trioid. If the operations of a trioid coincide, then the trioid becomes a semigroup.

Recall that a nonempty set T equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1) - (T3)$ is called a dimonoid (see, for example, [4, 5]).

Let $(T, \perp)$ be an arbitrary semigroup. Define operations $\dashv$ and $\vdash$ on T by

$$x \dashv y = x, \quad x \vdash y = y$$

for all $x, y \in T$.

Proposition 1. *([8], Proposition 10). $(T, \dashv, \vdash, \perp)$ is a trioid.*

The trioid $(T, \dashv, \vdash, \perp)$ will be denoted by $T_{lr}^{\perp}$.

Other examples of trioids can be found in [1, 6–8].

A commutative idempotent semigroup is called a semilattice.

Lemma 1. *([7], Lemma 1). The operations of a trioid $(T, \dashv, \vdash, \perp)$ coincide if $(T, \dashv)$ is a semilattice.*

Let $X = \{1, 2, 3\}$. For every pair $(x, y) \in X \times X$ let $T^{(x,y)} = (T, *_x, *_y)$ be an ordered triple, where T is a nonempty set and $*_x$, $*_y$ are binary operations on T. Let

$$B = \{(1, 1), (2, 2), (3, 3), (1, 2)\} \subset X \times X.$$

The following lemma describes connections between trioids and dimonoids.

Lemma 2. *For any trioid $(T, *_1, *_2, *_3)$ the algebra $T^{(x,y)}$, $(x, y) \in X \times X$, is a dimonoid if $(x, y) \in B$. There exists some trioid $(T, *_1, *_2, *_3)$ for which the algebra $T^{(x,y)}$, $(x, y) \in X^2 \backslash B$, is not a dimonoid.*

Proof. Let $(T, *_1, *_2, *_3)$ be a trioid. It is easy to see that the algebras $T^{(1,1)}$, $T^{(2,2)}$, $T^{(3,3)}$ and $T^{(1,2)}$ are dimonoids.

Now we shall prove the second part of the lemma.

Let $F[A]$ be the free semigroup on a set A and $F[A]_{lr}^{\perp}$ be a triod (see Proposition 1) such that $\perp$ is the concatenation on $F[A]$. Assume $(T, *_1, *_2, *_3) = F[A]_{lr}^{\perp}$ and show that for any $(x, y) \in X^2 \backslash B$ the algebra $T^{(x,y)}$ is not a dimonoid.

Let $w, u, \omega \in T^{(x,y)}$.

For $T^{(1,3)}$ check the axiom $(T3)$:

$$(w *_1 u) *_3 \omega = w *_3 \omega = w\omega \neq wu\omega = w *_3 (u *_3 \omega).$$

As the axiom $(T3)$ does not hold, then $T^{(1,3)}$ is not a dimonoid.

For $T^{(2,1)}$, $T^{(2,3)}$, $T^{(3,1)}$ and $T^{(3,2)}$ check the axiom $(T1)$.

For $T^{(2,1)}$ we have

$$(w *_2 u) *_2 \omega = \omega \neq u = w *_2 u = w *_2 (u *_1 \omega\ ).$$

For $T^{(2,3)}$:

$$(w *_2 u) *_2 \omega = \omega \neq u\omega = w *_2 (u *_3 \omega).$$

For $T^{(3,1)}$:

$$(w *_3 u) *_3 \omega = wu\omega \neq wu = w *_3 (u *_1 \omega).$$

For $T^{(3,2)}$:

$$(w *_3 u) *_3 \omega = wu\omega \neq w\omega = w *_3 (u *_2 \omega).$$

The axiom $(T1)$ does not hold for all fourth cases, so $T^{(2,1)}$, $T^{(2,3)}$, $T^{(3,1)}$ and $T^{(3,2)}$ are not dimonoids. □

The notion of a triband of subtrioids was introduced and investigated in [7]. Recall this definition.

A trioid $(T, \dashv, \vdash, \perp)$ is called an idempotent trioid or a triband if $x \dashv x = x \vdash x = x \perp x = x$ for all $x \in T$. If $\varphi : S \to M$ is a homomorphism of trioids, then the corresponding congruence on S will be denoted by Δ_φ.

Let S be an arbitrary trioid, J be some idempotent trioid and

$$\alpha : S \to J : x \mapsto x\alpha$$

be a homomorphism. Then every class of the congruence Δ_α is a subtrioid of the trioid S, and the trioid S itself is a union of such trioids $S_\xi,\ \xi \in J$ that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|(x,t) \in \Delta_\alpha\},$$

$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon}, \quad S_\xi \perp S_\varepsilon \subseteq S_{\xi \perp \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a triband of subtrioids (or S is a triband J of subtrioids $S_\xi,\ \xi \in J$). If J is a band (=idempotent semigroup), then we say that S is a band J of subtrioids $S_\xi,\ \xi \in J$. If J is a commutative band, then we say that S is a semilattice J of subtrioids $S_\xi,\ \xi \in J$.

Observe that the notion of a triband of subtrioids generalizes the notion of a diband of subdimonoids [5] and the notion of a band of semigroups [11].

Examples of trioids which are decomposed into a triband of subtrioids can be found in [7].

3 Main results

In this section we describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of s-simple subtrioids.

Let $(T, \dashv, \vdash, \perp)$ be an arbitrary dimonoid. Yamada introduced the notion of a P-subsemigroup of an arbitrary semigroup (see [9]). We denote by Ω the collection of all P-subsemigroups of $(T, \dashv)$ and by $T_\alpha, T_\beta, ...$ the elements of Ω.

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp)/_\rho$ coincide and it is a semilattice, then we say that ρ is a semilattice congruence.

For every subset Γ of Ω define a relation $\Gamma_\dashv$ on $(T, \dashv, \vdash, \perp)$ by

$$a\Gamma_\dashv b \text{ if and only if}$$
$$\{(x, y)|x \dashv a \dashv y \in T_\alpha\} = \{(x, y)|x \dashv b \dashv y \in T_\alpha\}$$

for every $T_\alpha \in \Gamma$.

Theorem 1. *The relation $\Gamma_\dashv$ on any trioid $(T, \dashv, \vdash, \perp)$ is a semilattice congruence. Conversely, any semilattice congruence on $(T, \dashv, \vdash, \perp)$ can be obtained by this way.*

Proof. The fact that the relation $\Gamma_\dashv$ is a semilattice congruence on a dimonoid $(T, \dashv, \vdash)$ has been proved in [10]. Show that $\Gamma_\dashv$ is compatible concerning the operation $\perp$.

Let $a\Gamma_\dashv b$, $a, b, c \in T$. As $a \dashv c\Gamma_\dashv b \dashv c$, then

$$\{(x, y)|x \dashv (a \dashv c) \dashv y \in T_\alpha\} = \{(x, y)|x \dashv (b \dashv c) \dashv y \in T_\alpha\}$$

for every $T_\alpha \in \Gamma$. By the associativity of the operation $\dashv$ and the axiom $(T4)$ of a trioid we obtain

$$x \dashv (a \dashv c) \dashv y = ((x \dashv a) \dashv c) \dashv y =$$
$$= (x \dashv (a\perp c)) \dashv y = x \dashv (a\perp c) \dashv y,$$
$$x \dashv (b \dashv c) \dashv y = ((x \dashv b) \dashv c) \dashv y =$$
$$= (x \dashv (b\perp c)) \dashv y = x \dashv (b\perp c) \dashv y.$$

So, $a\perp c\Gamma_\dashv b\perp c$. Analogously, we can prove that $c\perp a\Gamma_\dashv c\perp b$. Thus, $\Gamma_\dashv$ is a congruence on $(T, \dashv, \vdash, \perp)$.

As $(T, \dashv)/_{\Gamma_\dashv}$ is a semilattice, then by Lemma 1 the operations of $(T, \dashv, \vdash, \perp)/_{\Gamma_\dashv}$ coincide and so, it is a semilattice.

The converse statement follows from [9] (see also [10]). □

Theorem 1 generalizes Yamada's theorem [9] about the structure of all semilattice congruences on an arbitrary semigroup and the description [10] of all semilattice congruences on an arbitrary dimonoid.

A trioid $(T, \dashv, \vdash, \perp)$ will be called s-simple if its least semilattice congruence coincides with the universal relation on T.

Theorem 2. *The relation* $\Omega_{\dashv}$ *on any trioid* $(T, \dashv, \vdash, \perp)$ *is the least semilattice congruence. Every trioid* $(T, \dashv, \vdash, \perp)$ *is a semilattice of* s*-simple subtrioids.*

Proof. By Theorem 1 $\Omega_{\dashv}$ is a semilattice congruence. If $a\Omega_{\dashv}b$, $a, b \in T$, then it is easy to see that $a\Gamma_{\dashv}b$ for any $\Gamma \subseteq \Omega$. So, $\Omega_{\dashv} \subseteq \Gamma_{\dashv}$.

Now we shall prove the second statement of the theorem.

Since $\Omega_{\dashv}$ is a congruence on $(T, \dashv, \vdash, \perp)$ and $(T, \dashv, \vdash, \perp)/_{\Omega_{\dashv}}$ is a semilattice, then

$$(T, \dashv, \vdash, \perp) \to (T, \dashv, \vdash, \perp)/_{\Omega_{\dashv}} : x \mapsto [x]$$

is a homomorphism ($[x]$ is a class of the congruence $\Omega_{\dashv}$ which contains x). From [10] it follows that every class A of the congruence $\Omega_{\dashv}$ is an s-simple dimonoid concerning operations $\dashv$ and $\vdash$. Hence we obtain s-simplicity of the subtrioid A of a trioid $(T, \dashv, \vdash, \perp)$. □

Theorem 2 generalizes Yamada's theorem [9] about the structure of the least semilattice congruence on an arbitrary semigroup and the description [10] of the least semilattice congruence on an arbitrary dimonoid.

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Received March 3, 2013